Note on Numbering of Equations:

In the lectures which follow, references to equations in the preceding reprint are indicated by a capital R followed by the equation number.

Lecture XII — RADIATION BY A PREDETERMINED CHARGE-CURRENT DISTRIBUTION

Not many problems of quantum electrodynamics are in any sense exactly soluble. But there does exist one simple, completely soluble problem which has considerable physical meaning. That is the problem of finding the photon field radiated by an electric current distribution which is essentially classical in nature. By "classical" in this case we mean that we may represent the current by a prescribed vector function of space and time, $\mathbf{j}(\mathbf{r}t)$.

Such a model clearly can not represent the process of radiation by an individual atom, since the atomic current is affected by radiation recoil in essentially unpredictable ways. The model may, however, be an excellent approximation for dealing with radiation by aggregates of atoms which are large enough to show statistically predictable behavior for the total current vector. Note that in saying this we are not at all ignoring the reaction of the radiation process back upon the current. All we require is that whatever the reaction is, it be predictable at least in principle (as the radiation resistance of an antenna is, for example). It seems likely that this model, when allowance is made for statistical uncertainties in the current distribution, will accurately account for the photon fields generated by most macroscopic sources.

The interaction Hamiltonian which describes the coupling of the quantized electromagnetic field to the current distribution takes the form

$$\mathcal{H}_I(t) = \frac{1}{c}\int \mathbf{j}(\mathbf{r}, t)\cdot \mathbf{A}(\mathbf{r}, t)\, d\mathbf{r}. \qquad (12.1)$$

The state vector of the field changes with time in the interaction representation, obeying the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}|t> = \mathcal{H}_I|t> . \qquad (12.2)$$

Now let us, as an abbreviation, introduce the operator $B(t)$ which is defined as

$$B(t) = \frac{1}{\hbar c}\int \mathbf{j}(\mathbf{r}, t)\cdot \mathbf{A}(\mathbf{r}, t)\, d\mathbf{r}. \qquad (12.3)$$

The operator $B(t)$ is simply a linear combination of values of the vector potential, and hence obeys a commutation relation of the same general type as the vector potential. In general $[B(t), B(t')]$ will be different from zero, but it is always an ordinary number.

Now the Schrödinger equation, Eq. (12.2), can be rewritten as

$$\frac{d}{dt}|t> = B(t)|t> . \qquad (12.4)$$

Because of the operator character of $B(t)$ the solution of this equation is not

$$\exp\left\{\int_{t_0}^{t} B(t')\,dt'\right\}\ |\ t_0> \qquad (12.5)$$

as it would be if $B(t)$ were an ordinary number. However because of the simple commutation relation obeyed by the B's this expression will turn out not to be quite as wrong as we might perhaps expect.

We know that the state $|t>$ at time t can be expressed by means of a unitary operator, $U(t, t_0)$, applied to the state $|t_0>$ as time t_0, i.e.,

$$|t> = U(t, t_0)|t_0> . \qquad (12.6)$$

The equations which determine $U(t, t_0)$ are evidently

$$\frac{d}{dt}U(t, t_0) = B(t)\,U(t, t_0) \qquad (12.7)$$

and the initial condition $U(t_0, t_0) = 1$.

In order to solve for the operator U let us begin by dividing the time interval $(t_0 - t)$ into sub-intervals of length Δt extending between the times $t_j = t_0 + j\Delta t$, where j is an integral. We may then reach the solution of Eq. (12.7) through a simple limiting process. We assume that the operator $B(t)$ is constant in value during each of the sub-intervals of time and allow its value to change at the times t_j. A rather fanciful picture of this variation is shown in the "graph" of the operator B versus time given by Fig. 11.

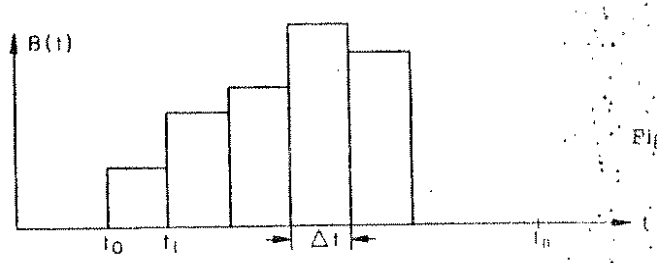

Figure 11

Since the operator B is constant in each of the sub-intervals, we can easily integrate the differential equation (12.7) for the individual sub-intervals. If $B(t)$ takes on the value B_j in the interval from t_{j-1} to t_j then we evidently have

$$U(t_j, t_{j-1}) = e^{B_j \Delta t} . \qquad (12.8)$$

Hence the transformation operator which corresponds to a succession of sub-intervals must be

$$U(t_n, t_0) = e^{B_n \Delta t}\, e^{B_{n-1}\Delta t} \cdots e^{B_1 \Delta t} . \qquad (12.9)$$

Now we can use the familiar theorem for multiplication of exponentials, Eq. (R 3.20) to evaluate the product. For n = 2, for example we have

$$U(t_2, t_0) = e^{B_2 \Delta t}\, e^{B_1 \Delta t} = \exp\left\{(B_1 + B_2)\Delta t + \frac{1}{2}[B_2, B_1](\Delta t)^2\right\} . \qquad (12.10)$$

The repetition of similar multiplications clearly leads to

$$U(t_n, t_0) = \exp\left\{\sum_{j=1}^{n} B_j \Delta t + \frac{1}{2}\sum_{j>k}[B_j, B_k](\Delta t)^2\right\} , \qquad (12.11)$$

which is an exact solution as long as $B(t)$ has the discontinuous time variation we have assumed.

We may consider the case in which the operator $B(t)$ varies continuously with time to be the limit in which $\Delta t \to 0$, i.e., we assume $t_n = t$ remains fixed and let $n \to \infty$. In that limit Eq. (12.12) becomes the general solution

$$U(t, t_0) = \exp\left\{\int_{t_0}^{t} B(t')dt' + \frac{1}{2}\int_{t_0}^{t}dt' \int_{t_0}^{t'}dt'' [B(t'), B(t'')]\right\} . \qquad (12.12)$$

If we compare this solution with the expression (12.5), which was reached by naively ignoring the operator character of B(t), we may see that the difference lies only in the addition of the term

$$\frac{1}{2}\int_{t_0}^{t}dt'\int_{t_0}^{t'}dt''\ [B(t'),\ B(t'')] \tag{12.13}$$

to the exponent. The commutator in this integral is an ordinary number and, in fact, a purely imaginary one. Hence the solution (12.12) only differs from (12.5) by a time dependent phase factor. If we let $i\varphi(t)$ represent the integral (12.14), then we may write the transformation operator as

$$\begin{aligned} U(t,t_0) &= \exp\left\{\int_{t_0}^{t}B(t')\,dt' + i\varphi(t)\right\} \\ &= \exp\left\{\frac{1}{\hbar c}\int_{t_0}^{t}\mathbf{j}(\mathbf{r},t')\cdot\mathbf{A}(\mathbf{r},t')\,dt'\,d\mathbf{r} + i\varphi(t)\right\}. \end{aligned} \tag{12.14}$$

Although the phase function $\varphi(t)$ is not altogether lacking in physical interest, (there is information contained in it, for example, on the interaction energy of the current and field) it does not have any influence on the calculation of density operators for the field, i.e., if the density operator has the initial value $\rho(t_0)$, then its value at time t is

$$\rho(t) = U(t,t_0)\,\rho(t_0)\,U^{\dagger}(t,t_0), \tag{12.15}$$

and we see immediately that the phase factor cancels.

If in particular the initial state is the vacuum state

$$|\,t_0\rangle = |\,\mathrm{vac}\rangle, \tag{12.16}$$

then at time t we have

$$e^{-i\varphi(t)}\,|t\rangle = \exp\left\{\frac{1}{\hbar c}\int_{t_0}^{t}\mathbf{j}(\mathbf{r},t')\cdot\mathbf{A}(\mathbf{r},t')\,d\mathbf{r}dt'\right\}|\,\mathrm{vac}\rangle. \tag{12.17}$$

Now if we introduce the expansion of the operator A in normal modes, Eq. (R 2.10), we see that the unitary operator which is applied to the vacuum state on the right side of Eq. (12.17) is simply a product of displacement operators which take the form

$$D_k(\alpha_k) = \exp\left[\alpha_k a_k^{\dagger} - \alpha_k^{*} a_k\right]. \tag{12.18}$$

More precisely, if we define the set of time-dependent amplitudes

$$\alpha_k(t) = \frac{1}{(2\hbar\omega_k)^{1/2}}\int_{t_0}^{t}\mathbf{j}(\mathbf{r},t')\cdot\mathbf{u}_k^{*}(\mathbf{r},t')\,e^{-i\omega_k t'}\,d\mathbf{r}dt', \tag{12.19}$$

then Eq. (12.17) may be rewritten as

$$e^{-i\varphi(t)}\,|t\rangle = \prod_k D_k(\alpha_k(t))\,|\,\mathrm{vac}\rangle. \tag{12.20}$$

It is clear from this result that a prescribed current distribution, radiating into the vacuum, always brings the field to a coherent state

$$e^{-i\varphi(t)}\,|t\rangle = |\{\alpha_k(t)\}\rangle. \tag{12.21}$$

More generally, if the field is initially in an arbitrary coherent state its state remains coherent under the influence of the current distribution.

The solution to the radiation problem we have found takes accurate account of the quantum mechanical properties of the field. It is related, however, in a simple way to the solution of the corresponding classical problem. The amplitudes $\alpha_k(t)$ are simply related to the time-dependent mode amplitudes for the classically radiated field through Eq. (8.22).

The density operator at time t which corresponds to the coherent state (12.21) is simply

$$\rho(t) = |\{\alpha_k(t)\}\rangle\langle\{\alpha_k(t)\}|, \tag{12.22}$$

which may be written in the P-representation as

$$\rho(t) = \int P(\{\beta_k\})\,|\{\beta_k\}\rangle\langle\{\beta_k\}|\prod_k d^2\beta_k, \tag{12.23}$$

by making use of the P-function

$$P(\{\beta_k\}) = \prod_k \delta^{(2)}(\beta_k - \alpha_k(t)). \tag{12.24}$$

The calculations we have carried out have dealt with a predetermined current distribution, i.e., one which behaves in a way which is in principle predictable. But in practice, of course, we may lack the information necessary to make such predictions and may have to resort to a statistical description of the behavior of the current. In that case, since we do not know the current $\mathbf{j}(\mathbf{r}, t)$ at any given time, it becomes impossible to make an exact specification of the set of amplitudes $\alpha_k(t)$ through Eq. (12.19). The best we can do is to state that the coefficients α_k have a certain probability distribution $p(\{\alpha_k\}, t)$ at time t whose dispersion corresponds to whatever randomness is present. Then it is clear that the density operator can be written in the form

$$\rho = \int p(\{\alpha_k\},t)\,|\{\alpha_k\}\rangle\langle\{\alpha_k\}|\prod_k d^2\alpha_k, \tag{12.25}$$

which is a fairly general form for the P-representation, but one in which the function P is obviously always positive.

Density operators having the general form of Eq. (12.25) with $p(\{\alpha_k\}, t)$ positive may arise from a variety of sources (e.g., thermal radiators, discharge tubes, etc.). Hence it is interesting to note that our arguments indicate that we can always construct for these cases some sort of random classical current distribution which will lead to the same field, i.e., the same density operator.

Lecture XIII PHASE-SPACE DISTRIBUTIONS FOR THE FIELD

In classical mechanics we can specify the state of a system by giving the instantaneous values of all coordinates and momenta. The evolution of the system then follows uniquely from the equations of motion. It can be visualized by considering the n coordinates and n momenta of the system as the coordinates of a point in a 2n-dimensional space, the phase space. The point which represents a system in this space moves along a uniquely determined trajectory. This picture is easily adapted to the uses of classical statistical mechanics. There, since we are characteristically uncertain of the initial coordinates and momenta of the system, we can speak only of probability distributions $P_{cl}(p_1', \cdots p_n', q_1', \cdots, q_n')$ for these variables. Instead of following the motion of a single point through the phase space, we must follow the motion of a whole "cloud" of them representing an ensemble of similarly prepared systems. The expectation value of any function of the p_i' and q_i' can then be calculated by means of an integral, involving the probability P_{cl} as

a weight function.

There has been, since the earliest days of quantum mechanics, a prevailing temptation to use the same sort of phase space picture for the description of quantum mechanical uncertainties. We shall not attempt to discuss these representations here in much generality since our interests are confined to the electromagnetic field. From a dynamical standpoint, the oscillations of each mode of the field are those of a harmonic oscillator. It will be quite sufficient, for the present discussion, to confine our attention to a single mode. In that case, the classical phase space has only two dimensions, corresponding to the variables p' and q'. The phase point for a mode with energy E moves classically along the ellipse $p'^2 + \omega^2 q'^2 = 2E$. (The mass parameter is set equal to unity.)

A coherent state of the mode will exist corresponding to any complex eigenvalue we specify for the operator

$$a = (2\hbar\omega)^{-\frac{1}{2}} (\omega q + ip) . \qquad (13.1)$$

The amplitude α corresponding to the state $|\alpha\rangle$ may be written as

$$\alpha = (2\hbar\omega)^{-\frac{1}{2}} (\omega q' + ip') , \qquad (13.2)$$

where q' and p' are real numbers. Now we have shown in Section III of the reprinted paper that the state $|\alpha\rangle$ may be described by a wave packet which has minimum uncertainty and the mean coordinate q' and the mean momentum p'.

Furthermore if we use the Schrödinger picture and follow the motion of the state with time, we know that the state remains coherent at all times, and that its time-dependent amplitude is simply $\alpha e^{-i\omega t}$. The motion of the amplitude vector in the complex α-plane takes place on the circle $|\alpha| = \text{const.}$ which simply represents an ellipse of the type noted earlier in the p', q' plane.

It is clear that the complex α-plane is simply a species of two-dimensional phase space. One therefore inevitably feels a great temptation to think of the coherent state wave packets in terms of probability "clouds" whose centers move on circular paths. Such an image, however, is an intrinsically classical one. In quantum mechanics the observables p and q are not simultaneously measureable (with more than limited accuracy), and therefore a certain lack of meaning, or at best an arbitrariness of meaning characterizes any attempt to speak of a joint probability distribution for the variables p' and q'. We can, of course, speak of the distribution of either variable in precisely defined terms, but these are alternative descriptions of the oscillator rather than a way of dealing with p' and q' simultaneously.

The P-representation of the density operator, which we introduced in the reprinted paper, can often be regarded as defining something at least comparable to a phase space distribution. The complex α-plane on which the P-function is defined, is indeed a species of phase space. Furthermore as we have noted in the paper, the P-function has a number of properties in common with probability distributions. However, as we have also seen, the function may take on negative values, and behave in singular ways which are altogether unlike those of a probability density. There is nothing inconsistent about such strange behavior because the function is not accessible to measurement as a joint probability distribution.

From the standpoint of similarity to classical theory, the function $P(\alpha)$ is simply one of a class of functions which possess, by definition, some of the properties of a phase space distribution and then inevitably lack others. We will discuss some other examples of such functions, which are perhaps best called quasi-probability densities, later in the lecture, and show their relation to the P-representation. First, however, let us turn to the question of how generally applicable the P-representation is.

THE P-REPRESENTATION AND THE MOMENT PROBLEM

Although it is clear, from the examples given in the reprinted paper, that the P-representation of the density operator is capable of representing a fairly broad variety of fields, no effort was made there to characterize that class of fields. Sudarshan has stated in a brief note,[1] however, that a "diagonal" representation of the density operator in terms of the coherent states may be used to represent an arbitrary field. He has given an explicit construction for the weight function of such a representation as an infinite sum of arbitrarily high-order derivatives of a delta function. He has said that, as a consequence of this construction, "the description of statistical states of a quantum mechanical system ... is completely equivalent to the description in terms of classical probability distributions."

The way in which Sudarshan's construction for the function $P(\alpha)$ may be reached is as follows: we consider the matrix elements of the density operator in the n-quantum state representation as known and note that, according to Eq. (R.7.12), these matrix elements are the complex moments,

$$\langle n|\rho|m\rangle = (n!\, m!)^{-\frac{1}{2}} \int P(\alpha)\, (\alpha^*)^m\, \alpha^n\, d^2\alpha$$

of the weight function $P(\alpha)$. We then consider this sequence of equations for all n and m to define a species of two-dimensional moment problem, i.e., we seek a function $P(\alpha)$ which has the correct matrix of moments. The general problem when stated thus becomes a notoriously difficult one, and one which need not, for arbitrary matrix elements $\langle n|\rho|m\rangle$, have a reasonable solution of any sort. Sudarshan's solution corresponds to taking advantage of some remarkable properties of the delta function and its derivatives which are perhaps most easily illustrated in a one-dimensional context.

Let us suppose that we are given the problem of finding a function f(x) on the interval $-\infty < x < \infty$ which has a specified set of moments M_n, i.e., we have

$$\int_{-\infty}^{\infty} f(x)\, x^n\, dx = M_n \qquad n = 0,\ 1,\ 2,\ \cdots \qquad (13.3)$$

If we write the j-th derivative of the delta function as

$$\delta^{(j)}(x) = \frac{d^j}{dx^j}\, \delta(x), \qquad (13.4)$$

then we observe that its moments are given by

$$\int_{-\infty}^{\infty} x^k\, \delta^{(j)}(x)\, dx = (-1)^j\, j!\, \delta_{jk} . \qquad (13.5)$$

In other words, each derivative of the delta function has one and only one non-vanishing moment. It would seem then that we can construct a "solution" of the general moment problem simply by writing

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\, M_n \delta^{(n)}(x) . \qquad (13.6)$$

The test of such a "solution" is ultimately whether or not it means anything.

Mathematicians have long noted that the delta function and its derivatives are not, strictly speaking, functions at all. More recently they have provided us with the theory of distributions (or generalized functions) as a means of dealing with these structures in more meaningful and rigorous terms.

Equations (13.4) and (13.5) assume a well-defined meaning in terms of distribution theory, but the theory shows that there is in general no useful meaning which can be attached to an infinite sum such as Eq. (13.6).

The "solution" exhibited by Sudarshan for the two dimensional moment problem takes the explicit form

$$P_s(\alpha) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(n!\ m!)^{\frac{1}{2}}}{(n+m)!} < n|\rho|m > \frac{1}{2\pi|\alpha|} e^{|\alpha|^2 - i(n-m)\theta} \left\{\left(-\frac{\partial}{\partial|\alpha|}\right)^{n+m} \delta(|\alpha|)\right\}, \tag{13.7}$$

where we have written $\alpha = |\alpha| e^{i\theta}$. Recently Holliday and Sage[2] have shown, by considering a simple example explicitly, that this expression cannot be construed as a generalized function of any sort. The example was that of the thermal density operator, and for it they showed that when the series (13.7) is multiplied by an extremeley well behaved test function (which vanishes outside a circle of finite radius in the α-plane), and the product is then integrated, the integral diverges. More recently, Cahill[3] has shown that whenever there is no upper bound to the number of quanta present, the series (13.7) will fail to be interpretable as a distribution (or a generalized function).

While these results indicate that Sudarshan's proposed representation is not, in general, meaningful, they leave open the larger question of the generality of the P-representation. They allow the possibility, in other words, that there might exist other constructions of the P-representation which are meaningful for all states of the field. Recently, however, D. Kastler and the lecturer[4] have demonstrated that the P-representation lacks the generality necessary to represent all states. They have shown in particular that there exist quantum states of the field for which it is not possible to find functions $P(\alpha)$ which are distributions. That means that all general results derived by using the P-representation must be qualified by the assumption that the representation exists.

A POSITIVE-DEFINITE "PHASE SPACE DENSITY"

We will now consider some other examples of quasiprobability functions, with different types of behavior and different degrees of usefulness. The first of these is the diagonal element, $< \alpha|\rho|\alpha >$, of the density operator. It is clear that $< \alpha|\rho|\alpha >$ is non-negative and that it is a well-defined function of α for all ρ. It is therefore a good deal closer to being a phase space density in its behavior than is $P(\alpha)$.

From the general expression for $R(\alpha^*, \beta)$ given by Eq. (R 6.1),

$$R(\alpha^*, \beta) = < \alpha|\rho|\beta > \exp\left\{\frac{1}{2}(|\alpha|^2 + |\beta|^2)\right\},$$

we have

$$< \alpha|\rho|\alpha > = R(\alpha^*, \alpha) e^{-|\alpha|^2}. \tag{13.8}$$

Hence, according to Eq. (R 6.6), the normalization condition on $< \alpha|\rho|\alpha >$ is

$$\frac{1}{\pi}\int < \alpha|\rho|\alpha > d^2\alpha = \frac{1}{\pi}\int R(\alpha^*, \alpha) e^{-|\alpha|^2} d^2\alpha = 1. \tag{13.9}$$

If the P-representation exists for the density operator ρ and has a weight function $P(\beta)$, we clearly have

$$\begin{aligned} < \alpha|\rho|\alpha > &= \int P(\beta) |< \alpha|\beta >|^2 d^2\beta \\ &= \int P(\beta) e^{-|\alpha - \beta|^2} d^2\beta. \end{aligned} \tag{13.10}$$

The function we are considering is simply a Gaussian convolution of the P-function.

We can use the function $< \alpha|\rho|\alpha >$ to calculate averages of products of operators which are in antinormal order in much the same way as products in normal order are averaged by means of the P-representation. Let us consider, for example, the average

$$\mathrm{Tr}\{\rho J(a) K(a^\dagger)\},$$

where J and K can be any functions of the annihilation and creation operators, respectively.

We can write this average as

$$\mathrm{Tr}\{K(a^\dagger)\rho J(a)\} = \frac{1}{\pi}\int \mathrm{Tr}\{|\alpha><\alpha| K(a^\dagger)\rho J(a)\} d^2\alpha$$

$$= \frac{1}{\pi}\int d^2\alpha < \alpha|K(a^\dagger)\rho J(a)|\alpha > = \frac{1}{\pi}\int < \alpha|\rho|\alpha > K(\alpha^*) J(\alpha) d^2\alpha. \tag{13.11}$$

Unfortunately we are not too often interested in evaluating the expectation values of antinormally ordered products of field operators. When the full set of modes of the field is considered such expectation values tend to contain divergent contributions from the vacuum fluctuations.

The function $< \alpha|\rho|\alpha >$ takes an interesting form for the n-th excited state of the oscillator. For these states we have

$$\rho_n = |n><n| = \frac{1}{n!}(a^\dagger)^n |0><0| a^n, \tag{13.12}$$

and therefore the result

$$< \alpha|\rho|\alpha > = \frac{1}{n!} |< \alpha|n >|^2 = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}. \tag{13.13}$$

This is an extremely well-behaved function, especially when we compare it with the analogous expression in the P-representation, which contains the 2n-th derivative of a delta function. The function $x^n e^{-x}$ has a maximum at $x = n$ and is quite sharply peaked there for large values of n. If we want to express the result (13.13) as a distribution in phase space we can substitute the expression (13.2) for α and write

$$< \alpha|\rho|\alpha > = \frac{1}{n!} \frac{(p'^2 + \omega^2 q'^2)^n}{(2\hbar\omega)^n} \exp\left\{-\frac{p'^2 + \omega^2 q'^2}{2\hbar\omega}\right\}. \tag{13.14}$$

This function evidently has its maximum value on the ellipse $(1/2)(p'^2 + \omega^2 q'^2) = n\hbar\omega$, that is to say on the classical orbit in phase space. It drops to zero on either side of the classical orbit while remaining positive everywhere.

Another example for which we can easily illustrate this "phase space density" is that of the Gaussian density operator. For that case we have

$$\begin{aligned} R(\alpha^*, \beta) &= \int P(\gamma) \exp\{\alpha^*\gamma + \beta\gamma^* - |\gamma|^2\} d^2\gamma \\ &= \frac{1}{\pi < n >}\int \exp\left\{-\frac{|\gamma|^2}{< n >} + \alpha^*\gamma + \beta\gamma^* - |\gamma|^2\right\} d^2\gamma \\ &= \frac{1}{\pi < n >}\int \exp\left\{-|\gamma|^2 \frac{1 + < n >}{< n >} + \alpha^*\gamma + \beta\gamma^*\right\} d^2\gamma. \end{aligned} \tag{13.15}$$

We can now make the substitution $\xi = \gamma\left\{\frac{1 + < n >}{< n >}\right\}^{\frac{1}{2}}$, which reduces the integral to the standard form

$$R(\alpha^*, \beta) = \frac{1}{\pi(1+\langle n\rangle)} \int \exp\left\{-|\xi|^2 + \left[\frac{\langle n\rangle}{1+\langle n\rangle}\right]^{\frac{1}{2}} (\alpha^* \xi + \beta \xi^*)\right\} d^2\xi$$

$$= \frac{1}{\pi(1+\langle n\rangle)} \exp\left\{\frac{\langle n\rangle}{1+\langle n\rangle}\, \alpha^*\beta\right\}. \qquad (13.16)$$

Hence we find

$$\langle \alpha|\rho|\alpha\rangle = R(\alpha^*, \alpha) e^{-|\alpha|^2}$$

$$= \frac{1}{\pi(1+\langle n\rangle)} \exp\left\{-\frac{|\alpha|^2}{1+\langle n\rangle}\right\}. \qquad (13.17)$$

If $\langle n\rangle$ goes to zero this expression becomes the Gaussian function $(1/\pi)\exp(-|\alpha|^2)$. In the same case the weight function $P(\alpha)$ would be a delta function at the origin. If $\langle n\rangle$ goes to infinity we have

$$\langle \alpha|\rho|\alpha\rangle \cong \frac{1}{\pi\langle n\rangle} e^{-\frac{|\alpha|^2}{\langle n\rangle}} \cong P(\alpha). \qquad (13.18)$$

In this limit $\langle \alpha|\rho|\alpha\rangle$ becomes equal to the P-distribution. That is so because the limit of large $\langle n\rangle$ is just the classical limit. There $P(\alpha)$ does indeed become interpretable as a classical phase space density, and the distinction between normally and antinormally ordered operators also vanishes, as a consequence of the correspondence principle.

WIGNER'S "PHASE SPACE DENSITY"

The Wigner distribution can be considered as the grandfather of all our quasi-probability functions. It exists and is well-behaved for all quantum states but seems to take on negative values without hesitation. We shall follow the approach used by Moyal[5] to define the Wigner distribution.

We begin by discussing a species of characteristic function which is defined as

$$X(\mu, \nu) = \langle e^{i(\mu p + \nu q)}\rangle, \qquad (13.19)$$

where p and q are operators. By using our theorem for the decompositon of exponentials, Eq. (R 3.20), we may write this expression as

$$X(\mu, \nu) = \mathrm{Tr}\left\{\rho\, e^{\frac{i\mu p}{2}}\, e^{i\nu q}\, e^{\frac{i\mu p}{2}}\right\} \qquad (13.20)$$

If we restrict consideration to a pure state, use the coordinate representation, and recall the interpretation of exponential functions of the momentum as coordinate displacement operators, we may rewrite Eq. (13.20) as

$$X(\mu, \nu) = \int \psi^*(q'' - \tfrac{\mu\hbar}{2})\, e^{i\nu q''}\, \psi(q'' + \tfrac{\mu\hbar}{2})\, dq'', \qquad (13.21)$$

where $\psi(q')$ is the wave function of the pure state. The Wigner function is then the Fourier transform of this characteristic function

$$W(p', q') \equiv \frac{1}{(2\pi)^2} \int \exp\{-i(\mu p' + \nu q')\} X(\mu, \nu)\, d\mu\, d\nu$$

$$= \frac{1}{(2\pi)^2} \int \exp\{-i(\mu p' + \nu q')\} \int \psi^*(q'' - \tfrac{\mu\hbar}{2})\, e^{i\nu q''} \times$$

$$\psi(q'' + \tfrac{\mu\hbar}{2})\, dq''\, d\mu\, d\nu$$

$$= \frac{1}{2\pi} \int e^{-i\mu p'} \int \psi^*(q'' - \tfrac{\mu\hbar}{2})\, \delta(q' - q'')\, \psi(q'' + \tfrac{\mu\hbar}{2})\, dq''\, d\mu$$

$$= \frac{1}{2\pi} \int \psi^*(q' - \tfrac{\mu\hbar}{2})\, e^{-i\mu p'}\, \psi(q' + \tfrac{\mu\hbar}{2})\, d\mu. \qquad (13.22)$$

If we substitue $y = -\mu\hbar$ in the latter expression we derive the form of the distribution originally stated by Wigner,

$$W(p', q') = \frac{1}{2\pi\hbar} \int \psi^*(q' + \tfrac{1}{2}y)\, e^{\frac{ip'y}{\hbar}}\, \psi(q' - \tfrac{1}{2}y)\, dy. \qquad (13.23)$$

It is obvious that whenever we have a wave function we can derive a Wigner distribution from it. Thus the distribution always exists, but it is not necessarily positive. When we have a mixture of states we must of course take a suitably weighted average of (13.23) over all the states which occur. The normalization condition for $W(q', p')$ is

$$\int W(q', p')\, dp'\, dq' = \int \delta(\mu)\, \delta(\nu)\, X(\mu, \nu)\, d\mu\, d\nu$$

$$= X(0, 0)$$

$$= 1. \qquad (13.24)$$

To compare the Wigner distribution with the others we have discussed, it is useful to express it in terms of the creation and annihilation operators $a^\dagger$ and a. Then if we define a complex Fourier transform variable

$$\lambda = -\mu\left\{\frac{\hbar\omega}{2}\right\}^{\frac{1}{2}} + i\nu\left\{\frac{\hbar}{2\omega}\right\}^{\frac{1}{2}}, \qquad (13.25)$$

we may write the operator which occurs in the exponent of the characteristic function as

$$-i(\mu p + \nu q) = \lambda a^\dagger - \lambda^* a, \qquad (13.26)$$

and the characteristic function itself becomes

$$X(\mu, \nu) = \langle e^{\lambda a^\dagger - \lambda^* a}\rangle$$

$$= \mathrm{Tr}\left\{\rho e^{\lambda a^\dagger} e^{-\lambda^* a}\right\} e^{-\frac{1}{2}|\lambda|^2}$$

$$= \mathrm{Tr}\left\{\rho e^{-\lambda^* a} e^{\lambda a^\dagger}\right\} e^{\frac{1}{2}|\lambda|^2}. \qquad (13.27)$$

We can now use the normally ordered form to express the Wigner function in terms of the P-representation. If we assume that the density operator possesses a P-representation, the characteristic function is given by

$$X(\mu, \nu) = \int P(\beta) \langle \beta|\, e^{\lambda a^\dagger}\, e^{-\lambda^* a}\, |\beta\rangle\, e^{-\frac{1}{2}|\lambda|^2}\, d^2\beta$$

$$= \int P(\beta)\, \exp\left\{\lambda\beta^* - \lambda^*\beta - \tfrac{1}{2}|\lambda|^2\right\} d^2\beta. \qquad (13.28)$$

In calculating the Fourier transform of X, i.e., the Wigner function, it is convenient to use a linear combination of α and α^* in the exponent rather than a combination of the classical variables q' and p'. We therefore write

$$i(\mu p' + \nu q') = \lambda\alpha^* - \lambda^*\alpha \qquad (13.29)$$

and

$$d\mu\, d\nu = \frac{2}{\hbar}\, d^2\lambda \quad . \tag{13.30}$$

Then the Fourier transform becomes

$$W(q', p') = \frac{1}{(2\pi)^2}\int \mathrm{Tr}\{\rho e^{\lambda(a^\dagger - a^*)}\, e^{-\lambda^*(a-\alpha)}\}\, e^{-\frac{1}{2}|\lambda|^2}\, \frac{2}{\hbar}\, d^2\lambda$$

$$= \frac{1}{2\pi\hbar}\int P(\beta)\exp\{\lambda(\beta^* - \alpha^*) - \lambda^*(\beta - \alpha) - \tfrac{1}{2}|\lambda|^2\}\, d^2\lambda\, d^2\beta\,. \tag{13.31}$$

We can reduce this integral to a standard form by the substitution $\xi = \frac{\lambda}{\sqrt{2}}$ which leads to

$$W(q', p') = \frac{1}{\pi\hbar}\int P(\beta)\, \exp\{\sqrt{2}\,\xi(\beta^* - \alpha^*) - \sqrt{2}\,\xi^*(\beta - \alpha) - |\xi|^2\} d^2\xi\, d^2\beta$$

$$= \frac{1}{\pi\hbar}\int P(\beta)\, \exp\{-2|\beta - \alpha|^2\}\, d^2\beta. \tag{13.32}$$

It is sometimes convenient to think of the Wigner function more directly as a function of the complex variable α, and to change its normalization accordingly. We therefore recall that

$$d^2\alpha = \frac{1}{\{2\hbar\omega\}^{\frac{1}{2}}}\left\{\frac{\omega}{2\hbar}\right\}^{\frac{1}{2}} dp'\, dq' = \frac{dp'\, dq'}{2\hbar} \tag{13.33}$$

and define the function

$$W(\alpha) = 2\hbar\, W(p', q') \tag{13.34}$$

so that

$$\int W(\alpha)\, d^2\alpha = 1 \quad . \tag{13.35}$$

The Wigner function of complex argument is then given by

$$W(\alpha) = \frac{2}{\pi}\int P(\beta)\, e^{-2|\beta-\alpha|^2}\, d^2\beta \quad . \tag{13.36}$$

When we compare this expression with the one derived in the preceding section,

$$\langle \alpha|\rho|\alpha\rangle = \int P(\beta)\, e^{-|\beta-\alpha|^2}\, d^2\beta, \tag{13.37}$$

We see that both of these expressions are simply Gaussian convolutions of the P-distribution (when the latter exists). The quality which the Wigner distribution shares with the P-distribution, of becoming negative in places, would seem to be due to the fact that the averaging process expressed by Eq. (13.36) takes place over a radius which is $(\sqrt{2})^{-1}$ times smaller than that expressed by Eq. (13.37).

As an example, let us evaluate the Wigner distribution for a field described by a Gaussian density operator. For this case we have, according to Eq. (13.36),

$$W(\alpha) = \frac{2}{\pi^2\langle n\rangle}\int \exp\left\{-\frac{|\beta|^2}{\langle n\rangle} - 2|\beta - \alpha|^2\right\} d^2\beta$$

$$= \frac{2\, e^{-2|\alpha|^2}}{\pi^2\langle n\rangle}\int \exp\left\{-|\beta|^2\left(2 + \frac{1}{\langle n\rangle}\right) + 2(\beta^*\alpha + \alpha^*\beta)\right\} d^2\beta \quad . \tag{13.38}$$

We now use the substitution $\gamma = \left\{\frac{1 + 2\langle n\rangle}{\langle n\rangle}\right\}^{\frac{1}{2}}\beta$ to reduce the integral to the standard form

$$W(\alpha) = \frac{2\, e^{-2|\alpha|^2}}{\pi^2(2\langle n\rangle + 1)}\int \exp\left\{-|\gamma|^2 + 2\left\{\frac{\langle n\rangle}{2\langle n\rangle + 1}\right\}^{\frac{1}{2}}(\gamma^*\alpha + \alpha^*\gamma)\right\} d^2\gamma$$

$$= \frac{2}{\pi^2(2\langle n\rangle + 1)}\int \exp\left\{-\left|\gamma - 2\left\{\frac{\langle n\rangle}{2\langle n\rangle + 1}\right\}^{\frac{1}{2}}\alpha\right|^2\right\} d^2\gamma \times$$

$$\exp\left\{\left[\frac{4\langle n\rangle}{2\langle n\rangle + 1} - 2\right]|\alpha|^2\right\} . \tag{13.39}$$

The latter integral leads immediately to the result

$$W(\alpha) = \frac{2}{\pi(2\langle n\rangle + 1)}\, \exp\left\{-\frac{2}{2\langle n\rangle + 1}|\alpha|^2\right\} \quad . \tag{13.40}$$

Thus, the Wigner distribution also has the Gaussian shape. We consider again the two limiting cases $\langle n\rangle = 0$, for which

$$W(\alpha) = \frac{2}{\pi}\, e^{-2|\alpha|^2}\,, \tag{13.41}$$

and $\langle n\rangle \to \infty$, for which

$$W(\alpha) \cong \frac{1}{\pi\langle n\rangle}\, e^{-\frac{|\alpha|^2}{\langle n\rangle}} = P(\alpha) \quad . \tag{13.42}$$

The latter result is the one we anticipate for the correspondence limit.

The simple Gaussian form given by Eq. (13.40) may be used to derive the complete set of Wigner distributions for the n-quantum states. This is possible because the function (13.40) may be regarded as a generating function for the Wigner distributions. Let us consider, for a moment, the general case of a density operator which may be written in the form

$$\rho = (1 - x)\sum_{n=0}^{\infty} x^n\, |n\rangle\langle n|, \tag{13.43}$$

where x is an arbitrary parameter. If we let $W_n(\alpha)$ be the Wigner function for the n-th quantum state, then, as a consequence of the linearity of W in ρ, we must have

$$W(\alpha) = (1 - x)\sum_{n=0}^{\infty} x^n\, W_n(\alpha) \quad . \tag{13.44}$$

Now if we make the identification $x = \langle n\rangle/(1 + \langle n\rangle)$, it becomes clear from Eq. (R 8.10) that ρ given by Eq. (13.43) is simply the Gaussian density operator. We can therefore write Eq. (13.40) alternatively in terms of the variable x as

$$W(\alpha) = \frac{2(1 - x)}{\pi(1 + x)}\, \exp\left\{-2\left(\frac{1 - x}{1 + x}\right)|\alpha|^2\right\}$$

$$= \frac{2(1 - x)}{\pi(1 + x)}\exp\left\{\frac{x}{1 + x}\, 4|\alpha|^2\right\}\exp\{-2|\alpha|^2\}. \tag{13.45}$$

This rather complicated exponential is just the generating function for the Laguerre polynomials L_n. In more familiar notation the generating function reads as

$$\frac{\exp\left\{-\frac{\rho u}{1-u}\right\}}{1-u} = \sum_{n=0}^{\infty} L_n(\rho)\frac{u^n}{n!} \quad . \tag{13.46}$$

Hence Eq. (13.45) yields the expansion

$$W(\alpha) = (1-x)\frac{2}{\pi}\sum_{n=0}^{\infty} x^n \frac{(-1)^n}{n!} L_n(4|\alpha|^2)e^{-2|\alpha|^2} \quad . \tag{13.47}$$

The Wigner function for the n-th excited state of the oscillator may thus be identified as

$$W_n(\alpha) = \frac{2}{\pi}\frac{(-1)^n}{n!} L_n(4|\alpha|^2)\, e^{-2|\alpha|^2} \quad . \tag{13.48}$$

These functions have quite a wiggly behavior in the complex phase plane. The n-th function has nodes on n concentric circles.

For the first two states we have, more explicitly,

$$W_0(\alpha) = \frac{2}{\pi}e^{-2|\alpha|^2} = \frac{2}{\pi}\exp\left\{-\frac{p'^2+\omega^2 q'^2}{2\hbar\omega}\right\} \tag{13.48}$$

$$W_1(\alpha) = \frac{2}{\pi}(4|\alpha|^2 - 1)\, e^{-2|\alpha|^2} \quad . \tag{13.49}$$

The function $W_1(\alpha)$ is sketched in Fig. 12.

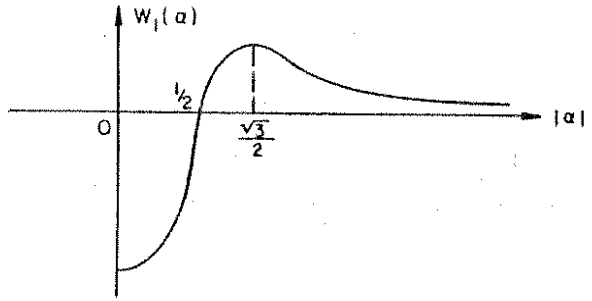

Figure 12

Its maximum lies at the radius $\alpha = \sqrt{3/2}$.

Each of the functions we have considered (the P-function, the function, $\langle -\alpha|\rho|\alpha\rangle$, and the Wigner distribution) has its particular advantages. It should, however, be clear from the preceding discussions that we can construct numerous other such functions, each with virtues of its own. An element of arbitrariness underlies all such discussions of phase space distributions.

Note added in proof: In a recent preprint, Klauder, McKenna, and Currie confirm the conclusion that no useful weight function P need exist for arbitrary density operators. To minimize this difficulty they express matrix elements of the density operator through a limiting procedure involving an infinite sequence of operators expressed as P-representations. This procedure, however, does not preserve the most useful property of the P-representation, the reduction of statistical averages to simple integrals over the complex α-plane.

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Lecture XIV

CORRELATION FUNCTIONS AND QUASIPROBABILITY DISTRIBUTIONS

In this lecture and in the ones which follow we shall begin to discuss applications of our formalism in somewhat more concrete terms. As a first step in that direction it will be useful to amplify several of the points which are stated rather briefly in the last section of the reprinted paper.

Let us suppose that the electromagnetic field is in a pure coherent state which we denote by $|\{\alpha_k\}>$. Then $|\{\alpha_k\}>$ is an eigenstate of the operator $E^{(+)}$,

$$E^{(+)}(rt)\,|\{\alpha_k\}> = \mathcal{E}(rt\,\{\alpha\})\,|\{\alpha_k\}>. \tag{14.1}$$

and the corresponding eigenvalue function $\mathcal{E}$, is a linear form in the variables $\{\alpha_k\}$, i.e., we have

$$\mathcal{E}(rt\,\{\alpha_k\}) = i\sum_k \left(\frac{\hbar\omega_k}{2}\right)^{\frac{1}{2}} u_k(r)\, e^{-i\omega_k t}\,\alpha_k. \tag{14.2}$$

The corresponding field is fully coherent since the correlation functions for all orders n fall into the factorized form

$$G^{(n)}_{\mu_1\cdots\mu_{2n}}(x_1\ldots x_{2n}) = \prod_{j=1}^{n}\mathcal{E}^*_{\mu_j}(x_j\{\alpha_k\})\prod_{j=n+1}^{2n}\mathcal{E}_{\mu_j}(x_j\{\alpha_k\}). \tag{14.3}$$

We have already noted that the term "coherence" is used frequently in the discussion of quantum mechanical problems of all sorts. Since the term is usually meant to imply that interference phenomena can take place, many of its uses are to be found in discussions of pure quantum mechanical states. Pure states, however, by no means exhaust the possibilities of securing interference. For most quantum mechanical systems there exist certain statistical mixtures of states which preserve essentially the same interference phenomena as are found for pure states. It is easy to exhibit these mixtures for the case of the electromagnetic field and to show that they may correspond to fields which are fully coherent in the sense of Eq. (14.3).

Instead of considering the field which corresponds to the set of amplitudes $\{\alpha_k\}$, let us consider the field corresponding to a set $\{\alpha_k'\}$ which we obtain by multiplying each of the coefficients α_k by a phase factor, $e^{i\phi}$, which is the same for all modes. If we have

$$\alpha_k' = e^{i\phi}\alpha_k, \tag{14.4}$$

then, since the eigenvalue function, $\mathcal{E}$, is linear, we must have

$$\mathcal{E}_\mu(rt\,\{\alpha_k'\}) = e^{i\phi}\,\mathcal{E}_\mu(rt\{\alpha_k\}). \tag{14.5}$$

Because the phase factors cancel when we construct the correlation functions, it is clear that the altered state of the field leads to the same set of correlation functions (14.3) as the original state. This invariance property, which is implicit in our definition of the correlation functions, means that we secure the same correlation functions not only for pure states corresponding to different values of the phase ϕ, but for arbitrary mixtures of such states as well.

Let us suppose that $\mathcal{L}(\phi)$ is a function which satisfies the normalization condition

$$\int_0^{2\pi}\mathcal{L}(\phi)\,d\phi = 1. \tag{14.6}$$

Then we may construct a density operator

$$\rho = \int_0^{2\pi} \mathcal{L}(\phi) |\{a_k e^{i\phi}\}\rangle\langle\{a_k e^{i\phi}\}| d\phi, \tag{14.7}$$

which represents mixtures of states with different values of the overall phase ϕ. (Note that $\mathcal{L}(\phi)$ must also statisfy a positive definiteness condition analogous to Eq. (R 7.9).) All such mixtures, i.e. all choices of $\mathcal{L}(\phi)$, lead to precisely the set of correlation functions (14.3); hence all such mixed states correspond to fully coherent fields.

It is most important, from a practical standpoint, that our definitions permit these mixed states to correspond to coherent fields. Our a priori knowledge of the state of high frequency fields usually contains no information about the overall phase ϕ. An ensemble of experiments performed with such fields must then be described by using a density operator of the form (14.7) with the special choice,

$$\mathcal{L}(\phi) = \frac{1}{2\pi}, \tag{14.8}$$

which represents our total ignorance of the phase. The indefinite character of this phase does not influence any of the interference intensities we have discussed thus far. It must therefore have no bearing on the coherence properties of a field. Our definition of coherence would hardly be very useful physically if it did not allow the appropriate mixed states as well as pure ones to be coherent.

FIRST ORDER CORRELATION FUNCTIONS FOR STATIONARY FIELDS

Virtually all of the famous experiments of optics may be described in terms of the first order correlation function for stationary light beams. Let us begin the evaluation of such a correlation function by using the normal mode expansion for the field operators to write it in the form

$$G^{(1)}_{\mu\nu}(rt, r't') = \frac{1}{2} \sum_{k,k'} \hbar(\omega_k \omega_{k'})^{\frac{1}{2}} \mathrm{Tr}\{\rho\, a_k^\dagger a_{k'}\} \times u_{k\mu}^*(r)\, u_{k'\nu}(r')\, e^{i(\omega_k t - \omega_{k'} t')}. \tag{14.9}$$

To evaluate the statistical averages $\mathrm{Tr}\{\rho a^\dagger_k a_{k'}\}$ we first note that these will always vanish when the modes k and k' are non-degenerate. We may prove that they vanish in this case by recalling that for stationary fields ρ commutes with the field Hamiltonian $\mathcal{H}_0$. Thus we have, for example,

$$\rho = e^{-\frac{i}{\hbar}\mathcal{H}_0 t}\, \rho\, e^{\frac{i}{\hbar}\mathcal{H}_0 t}, \tag{14.10}$$

for all values of the parameter t. If we substitute the latter form for the operator into the expression for the desired trace we find

$$\mathrm{Tr}\{\rho a^\dagger_k a_{k'}\} = \mathrm{Tr}\{\rho\, e^{\frac{i}{\hbar}\mathcal{H}_0 t} a_k^\dagger a_{k'} e^{-\frac{i}{\hbar}\mathcal{H}_0 t}\} = \mathrm{Tr}\{\rho a_k^\dagger a_{k'}\} e^{i(\omega_k - \omega_{k'})t}. \tag{14.11}$$

Since the trace is independent of the parameter t, it must vanish whenever $\omega_k \neq \omega_{k'}$.

For the case of two different but degenerate modes, k and k', on the other hand, the quantity $\mathrm{Tr}(\rho\, a_k^\dagger a_{k'})$ need not vanish. More generally, if there are N degenerate modes the corresponding averages $\mathrm{Tr}(\rho\, a_k^\dagger a_{k'})$ can be regarded as forming the elements of an N × N Hermitian matrix which is not, in general, diagonal. It is always possible to diagonalize this matrix, however, by means of a

linear transformation which amounts simply to a redefinition of the set of degenerate mode functions. For any stationary state of the field represented by a density operator ρ, in other words, there will exist some particular choice of mode functions $u_k(r)$ such that the matrix reduces to diagonal form, i.e. we have

$$\mathrm{Tr}(\rho\, a_k^\dagger a_{k'}) = \langle n_k \rangle \delta_{kk'}, \tag{14.12}$$

where $\langle n_k \rangle$ is the mean occupation number of the k-th mode.

The convenience of working with particular choices of degenerate mode functions is easily illustrated by means of the polarization properties of light beams. For any plane wave state of a beam there are two degenerate polarization modes which are othogonal. If we were to choose a pair of plane polarization states as a basis, and were to describe a circularly polarized beam, for example, the quantities $\mathrm{Tr}(\rho a_k^\dagger a_{k'})$ would form a 2 × 2 matrix with four non-vanishing components. It is no surprise then that a more convenient choice of mode functions for that case consists of the two orthogonal circular polarizations. That choice reduces the matrix to one with only a single non-vanishing component.

Let us now return to our calculation of the first order correlation function for stationary fields. We see from Eqs. (14.12) and (14.9) that with a suitable choice of basis functions it is always possible to write the correlation function as an expansion of the form

$$G^{(1)}_{\mu\nu}(rt, r't') = \frac{1}{2} \sum_k \hbar\omega_k \langle n_k \rangle u^*_{k\mu}(r) u_{k\nu}(r') e^{i\omega_k(t-t')}, \tag{14.13}$$

which is determined simply by the set of average occupation numbers $\langle n_k \rangle$. An expansion of this type which is often useful is based on the set of plane wave modes of a large cubical volume of side L. These modes, whose functions $u_k(r)$ are given by Eq. (R 2.9), are so densely distributed in the space of the propagation vector k, when the volume of the system is large, that the sum over the states required in Eq. (14.13) may be replaced by the integral $(L/2\pi)^3 \int dk \ldots$. The expansion of the correlation function is then

$$G^{(1)}(rt, r't') = \frac{\hbar c}{2(2\pi)^3} \int \sum_{\lambda=1,2} e_\mu^{(\lambda)*} e_\nu^{(\lambda)} \langle n_{k,\lambda} \rangle k \times \exp\{-i[k\cdot(r - r') - \omega_k(t - t')]\}\, dk, \tag{14.14}$$

where λ is an index which labels the polarizations associated with propagation vector.

Let us suppose that the field consists of a well collimated light beam which is nearly monochromatic and is fully polarized. Then the mean occupation number $\langle n_{k,\lambda} \rangle$ will only take on non-vanishing values within a very small cell of k-space and, say, for $\lambda = 1$. Under these circumstances, if the magnitudes of $|r - r'|$ and $c|t - t'|$ remain small in comparison to the reciprocal dimensions of the volume in which $\langle n_{k,\lambda} \rangle$ differs from zero, it becomes possible to approximate the integral in Eq. (14.14) by neglecting the variation of the exponential in the integrand. If k_0 and ω_0 are the mean propagation vector and frequency of the beam we have

$$G^{(1)}_{\mu\nu}(rt, r't') \approx \frac{\hbar c}{2(2\pi)^3} N e_\mu^{(1)*} e_\nu^{(1)} e^{-i|k_0\cdot(r-r') - \omega_0(t-t')|}, \tag{14.15}$$

where

$$N = \int \langle n_{k,1} \rangle dk. \tag{14.16}$$

The light beam we have described is of just the sort most often used in interference experiments. It is also the kind most often referred to as "coherent" in the traditional terminology of optics. Now it is evident that by defining the field

$$\mathcal{E}(r,t) = \left\{\frac{\hbar c}{\mathcal{V}(2\pi)^3}\right\} N^{\frac{1}{2}} \hat{e}^{(1)} e^{i(k_0 \cdot r - \omega_0 t)} \tag{14.17}$$

we may write the expression (14.15) for the correlation function in the factorized form

$$G^{(1)}_{\mu\nu}(rt, r't') = \mathcal{E}_\mu^*(rt)\, \mathcal{E}_\nu(r't'). \tag{14.18}$$

Hence the field in question does indeed satisfy the condition for first order coherence. It is worth emphasizing, however, that the factorization in Eq. (14.18) is an approximate one which tends to be most accurate for points r', t' near r, t. The imperfect collimation and monochromaticity of the beam define finite ranges of the variables r - r' and t - t', i.e. coherence distances and a coherence time, within which the factorization condition is obeyed. These ranges can, in principle, be made arbitrarily large by improving the quality of the beam.

This example illustrates the sense in which the coherence conditions must usually be regarded as idealizations. Given the practical sorts of field sources at our disposal, we cannot expect that the field correlations they generate will obey the coherence conditions over infinite ranges of the coordinate variables (even though in the case of laser fields these conditions may be known to hold over tens of thousands of miles).

CORRELATION FUNCTIONS FOR CHAOTIC FIELDS

A particularly important class of stationary fields, which arises whenever the source is essentially chaotic in nature, is one in which the weight function in the P-representation is a product of Gaussian factors, one for each mode. The density operator is then specified by

$$P(\{\alpha_k\}) = \prod_k \frac{1}{\pi \langle n_k \rangle} e^{-\frac{|\alpha_k|^2}{\langle n_k \rangle}}, \tag{14.19}$$

and it follows that all of the statistical properties of the field are determined by the set of average occupation numbers $\langle n_k \rangle$. The knowledge of this same set of numbers, on the other hand, is equivalent, according to Eq. (14.13), to specifying the first order correlation function for the field. There thus exists a fundamental sense in which the first order correlation function furnishes all the information we need for the description of fields specified by Gaussian weight functions. We may demonstrate this simplifying property more explicitly by showing that all of the higher order correlation functions for such fields can be expressed as sums of products of first order correlation functions.

In order to prove this theorem we shall construct a species of generating functional for the set of all correlation functions of the field. The essential tool for doing this is the operation of functional differentiation. If $F[\zeta(x)]$ is a functional of $\zeta(x)$, i.e. a function of the set of values of $\zeta(x)$ for all x, then we define its functional derivative with respect to $\zeta(x_0)$ to be

$$\frac{\delta F}{\delta \zeta(x_0)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{F[\zeta(x) + \epsilon \delta^{(4)}(x - x_0)] - F[\zeta(x)]\right\}, \tag{14.20}$$

where $\delta^{(4)}$ is a four-dimensional (space-time) delta function. As an illustration, if we apply this definition to an integral operator of the form

$$F = \int \zeta(x)\, E^{(-)}(x)\, d^4x, \tag{14.21}$$

we find

$$\frac{\delta F}{\delta \zeta(x_0)} = \int \delta^{(4)}(x - x_0)\, E^{(-)}(x)\, d^4x = E^{(-)}(x_0). \tag{14.22}$$

Now, let us define the generating functional

$$\Xi[\zeta(x), \eta(x)] = \mathrm{Tr}\left\{\rho\, e^{\int \zeta(x) E^{(-)}(x) d^4x}\, e^{\int \eta(x') E^{(+)}(x') d^4x'}\right\}, \tag{14.23}$$

which depends upon two independent functions $\zeta(x)$ and $\eta(x)$ and is the trace of a normally ordered product. Then we easily see that the functional derivatives of this expression, evaluated for $\zeta(x) = \eta(x) = 0$, are the correlation functions of the field; i.e. we have

$$\frac{\delta^2}{\delta\zeta(x_1)\,\delta\eta(x_2)} \Xi\Big|_{\zeta=\eta=0} = \mathrm{Tr}\left\{\rho\, E^{(-)}(x_1)\, E^{(+)}(x_2)\right\} = G^{(1)}(x_1, x_2), \tag{14.24}$$

and more generally

$$\frac{\delta^{2n}}{\delta\zeta(x_1)\cdots\delta\zeta(x_n)\,\delta\eta(x_{n+1})\cdots\delta\eta(x_{2n})} \Xi\Big|_{\zeta=\eta=0} = G^{(n)}(x_1 \cdots x_n, x_{n+1} \cdots x_{2n}). \tag{14.25}$$

(The tensor indices which have been suppressed in these expressions may be restored by considering each coordinate x to specify a component index as well as a position and a time, e.g. the function $\zeta(x)$ is actually a set of four functions $\zeta_\mu(r, t)$ for $\mu = 1, \cdots 4$, etc.)

It is convenient, at this point, to introduce the abbreviation

$$e(x, k) = i\left\{\frac{\hbar\omega_k}{2}\right\}^{\frac{1}{2}} u_k(r)\, e^{-i\omega_k t}, \tag{14.26}$$

which permits us to write the expansion of the operator $E^{(+)}$ in terms of the mode functions as

$$E^{(+)}(x) = \sum_k e(x, k)\, a_k. \tag{14.27}$$

Then when we use the P-representation for the density operator with the Gaussian weight function (14.19), the generating functional (14.23) may be written as

$$\Xi = \int e^{-\sum_k \frac{|\alpha_k|^2}{\langle n_k \rangle}} e^{\sum_k \int \zeta(x) e^*(x,k) \alpha_k d^4x} e^{\sum_k \int \eta(x') e(x',k) \alpha_k^* d^4x'} \prod_k \frac{d^2\alpha_k}{\pi \langle n_k \rangle}. \tag{14.28}$$

This multiple integral factors into a product of integrals, one for each mode k. If we introduce the pair of complex parameters

$$\beta_k = \int \zeta(x)\, e^*(x, k)\, d^4x$$

$$\gamma_k = \int \eta(x')\, e(x', k)\, d^4x', \tag{14.29}$$

the integral factor for the k-th mode takes the familiar form

$$\int \exp\left\{-\frac{|\alpha_k|^2}{\langle n_k\rangle} + \beta_k \alpha_k^* + \gamma_k \alpha_k\right\} \frac{d^2\alpha_k}{\pi\langle n_k\rangle} = \exp\{\beta_k \gamma_k \langle n_k\rangle\}. \quad (14.30)$$

Hence the generating functional is given by the product

$$\Xi = \prod_k \exp\{\beta_k \gamma_k \langle n_k\rangle\}$$
$$= \exp\left\{\int \zeta(x) \sum_k e^*(x, k)\, e(x', k) \langle n_k\rangle \eta(x') d^4x\, d^4x'\right\}. \quad (14.31)$$

Now, according to Eqs. (14.13) and (14.26), the first order correlation function for the field is given by the expansion

$$G^{(1)}(x, x') = \sum_k e^*(x, k)\, e(x', k) \langle n_k\rangle, \quad (14.32)$$

which is just the sum which occurs in the exponential function of Eq. (14.31). Hence the generating functional for the correlation functions of all orders may be expressed in terms of the first order correlation function as

$$\Xi[\zeta(x), \eta(x)] = \exp\left\{\int \zeta(x) G^{(1)}(x, x') \eta(x')\, d^4x\, d^4x'\right\}. \quad (14.33)$$

We may now derive explicit expressions for the higher order correlation functions by evaluating the appropriate functional derivatives. In particular the n-th derivative with respect to ζ may be written as

$$\frac{\delta^n}{\delta\zeta(x_1)\cdots\delta\zeta(x_n)} \Xi = \left\{\prod_{j=1}^n \int G^{(1)}(x_j, x')\eta(x') d^4x'\right\} \Xi. \quad (14.34)$$

To evaluate the n-th order correlation function we must next differentiate n times with respect to the function η. Since $\zeta(x)$ is finally to be set equal to zero it is easy to see that all of the terms which come from differentiating the factor Ξ on the right side of Eq. (14.34) with respect to η will finally vanish. Hence we have simply

$$\frac{\delta^{2n}}{\delta\zeta(x_1)\cdots\delta\zeta(x_n)\,\delta\eta(x_{n+1})\cdots\delta\eta(x_{2n})} \Xi \Big|_{\substack{\zeta=0\\ \eta=0}}$$
$$= \frac{\delta^n}{\delta\eta(x_{n+1})\cdots\delta\eta(x_{2n})} \prod_{j=1}^n \int G^{(1)}(x_j, x')\eta(x') d^4x'$$
$$= \sum_P \prod_{j=1}^n G^{(1)}(x_j, x_{P(n+j)}); \quad (14.35)$$

i.e., the derivative is a sum taken over the n! possible ways of permuting the set of coordinates $x_{n+1}, \cdots x_{2n}$. Since the derivative we have evaluated, according to Eq. (14.25), is the n-th order correlation function, we have finally

$$G^{(n)}(x_1 \cdots x_n, x_{n+1} \cdots x_{2n}) = \sum_P \prod_{j=1}^n G^{(1)}(x_j, x_{P(n+j)}). \quad (14.36)$$

The n-th order correlation function for Gaussian fields is just a symmetrical sum of products of first order correlation functions.

To illustrate this result for the second order correlation function we may write

$$G^{(2)}(x_1x_2, x_3x_4) = G^{(1)}(x_1x_3)G^{(1)}(x_2x_4) + G^{(1)}(x_1x_4)G^{(1)}(x_2x_3). \quad (14.37)$$

Now if the field in question possesses first order coherence, we may write the first order correlation function in the factorized form of Eq. (7.15). The two terms of Eq. (14.37) are then equal and we find

$$G^{(2)}(x_1x_2, x_3x_4) = 2\mathcal{E}^*(x_1)\mathcal{E}^*(x_2)\mathcal{E}(x_3)\mathcal{E}(x_4). \quad (14.38)$$

The second order correlation function factorizes, but because of the presence of the factor of 2, it does so in a way which precludes the possibility that the field has second or higher order coherence. The n-th order correlation function for such fields is evidently given by

$$G^{(n)}(x_1 \cdots x_{2n}) = n! \prod_{j=1}^n \mathcal{E}^*(x_j) \prod_{j=n+1}^{2n} \mathcal{E}(x_j). \quad (14.39)$$

QUASIPROBABILITY DISTRIBUTION FOR THE FIELD AMPLITUDE

Whenever the density operator for the field may be specified by means of the P-representation the function $P(\{\alpha_k\})$ plays a role analogous to that of a probability density for the individual mode amplitudes α_k. Of course when we make measurements upon a light beam, we are typically measuring not the individual amplitudes α_k, but the average values of various functions of the complex field strength eigenvalue, $\mathcal{E}(rt)$, which is a particular linear sum of the mode amplitudes,

$$\mathcal{E}(x, \{\alpha_k\}) = \sum_k e(x, k)\, \alpha_k. \quad (14.40)$$

To describe the fullest variety of such measurements which we can make at a single space-time point $x = (r, t)$, it is convenient to derive from $P(\{\alpha\})$ a species of reduced quasiprobability distribution for the complex field amplitude $\mathcal{E}(x, \{\alpha_k\})$. This distribution function for the field amplitude will be quite useful in discussing the origin of the photon correlation effect discovered by Hanbury Brown, and Twiss.

To illustrate the kinds of averages we frequently want to discuss, let us note that the average intensity of the field at the point x is

$$G^{(1)}(x, x) = \int P(\{\alpha_k\}) \left|\mathcal{E}(x\{\alpha_k\})\right|^2 \prod_k d^2\alpha_k, \quad (14.41)$$

and the average coincidence rate for the limiting case in which the two counters are placed at the same point and are sensitive at the same time is

$$G^{(1)}(x\,x,\, x\,x) = \int P(\{\alpha_k\}) \left|\mathcal{E}(x\{\alpha_k\})\right|^4 \prod_k d^2\alpha_k. \quad (14.42)$$

These are examples of a general class of averages which take the form

$$\int P(\{\alpha_k\})\, F(\mathcal{E}(x\{\alpha_k\})) \prod_k d^2\alpha_k \quad (14.43)$$

for suitably determined functions F. It is convenient now to separate the multidimensional integration over the complex amplitude parameters α_k into two steps, the integration over the subspace of the α_k-parameters in which the linear combination

$$\mathcal{E}(x\{\alpha_k\}) = \sum e(x, k)\alpha_k$$

remains constant, and then the further integration over the values this sum may take on. The first of these integrations is accomplished by defining the function

$$W(\mathcal{E}, x) = \int P(\{\alpha_k\})\, \delta^{(2)}\Big(\mathcal{E} - \sum_k e(x, k)\, \alpha_k\Big) \prod_k d^2\alpha_k. \quad (14.44)$$

We may then write the complete integral (14.43) in the form

$$\int P(\{\alpha_k\})\, F(\mathcal{E}(x\{\alpha_k\}))\prod_k d^2\alpha_k = \iint P(\{\alpha_k\})\,\delta^{(2)}\Big(\mathcal{E}-\sum_k e(x,k)\alpha_k\Big)\times F(\mathcal{E})\prod_k d^2\alpha_k\, d^2\mathcal{E}$$

$$= \int W(\mathcal{E}, x)\, F(\mathcal{E})\, d^2\mathcal{E}, \qquad (14.45)$$

where $d^2\mathcal{E} = d(\mathrm{Re}\,\mathcal{E})\, d(\mathrm{Im}\,\mathcal{E})$ is a real element of area in the complex field amplitude plane. The function $W(\mathcal{E}, x)$ defined by Eq. (14.44) evidently plays a role analogous to that of a probability distribution for the complex field amplitude at the space-time point x. Of course, since the function P from which it is derived is only a quasiprobability distribution, and is subject to all the restrictions mentioned in the last lecture, the same limitations will apply to the physical interpretation of the function $W(\mathcal{E}, x)$. It too can take on negative values, for example.

The function W furnishes a particularly simple description of fields which consist of many independently excited modes. Since the total field amplitude $\mathcal{E}$ is then the sum of a large number of independently distributed complex amplitudes proportional to the α_k, the distribution of the amplitude $\mathcal{E}$ will correspond to that of the endpoint of a many-step random walk in the complex plane. This distribution tends to take on a Gaussian form when the number of contributing modes is large, no matter how the mode amplitudes may be distributed individually. From a mathematical standpoint this argument differs hardly at all from the discussion of the central limit theorem given in Section VIII of the reprinted paper; i.e., the starting point, Eq. (14.44), becomes similar in structure to Eq. (R 8.1) when the function $P(\{\alpha_k\})$ is assumed to factorize into a product $\prod_k P_k(\alpha_k)$. As a slight generalization of the discussion given there we may let the individual mode excitations be non-stationary in character and have mean amplitudes

$$\int P_k(\alpha_k)\,\alpha_k\, d^2\alpha_k = \langle\alpha_k\rangle. \qquad (14.46)$$

Then by applying the central limit theorem, we find

$$W(\mathcal{E}, x) = \frac{1}{\pi\Sigma_k |e(x,k)|^2\{\langle|\alpha_k|^2\rangle - |\langle\alpha_k\rangle|^2\}}\times \exp\left\{-\frac{|\mathcal{E}-\Sigma_k e(x,k)\langle\alpha_k\rangle|^2}{\Sigma_k|e(x,k)|^2\{\langle|\alpha_k|^2\rangle-|\langle\alpha_k\rangle|^2\}}\right\}. \qquad (14.47)$$

If the mean amplitudes $\langle\alpha_k\rangle$ vanish, as they do for example in the case of stationary fields, we have

$$W(\mathcal{E}, x) = \frac{1}{\pi\Sigma_k|e(x,k)|^2\langle n_k\rangle}\exp\left\{-\frac{|\mathcal{E}|^2}{\Sigma_k|e(x,k)|^2\langle n_k\rangle}\right\} = \frac{1}{\pi G^{(1)}(x,x)}\, e^{-\frac{|\mathcal{E}|^2}{G^{(1)}(x,x)}}. \qquad (14.48)$$

To illustrate the use of this expression for $W(\mathcal{E}, x)$, let us calculate the n-th order correlation function with all arguments equal. By letting $F(\mathcal{E}) = |\mathcal{E}|^{2n}$ in Eq. (14.45) we find

$$G^{(n)}(x\cdots x) = \int W(\mathcal{E}, x)\,|\mathcal{E}|^{2n}\, d^2\mathcal{E}. \qquad (14.49)$$

For the Gaussian form of W given by Eq. (14.48), the latter integral is simply

$$G^{(n)}(x\cdots x) = n!\{G^{(1)}(x,x)\}^n. \qquad (14.50)$$

An important class of fields which obey the separability conditions we have assumed in deriving these results is that specified by the Gaussian density operators discussed earlier. For these fields, in fact, Eq. (14.50) follows directly from Eq. (14.36). But since we have not had to assume that the functions $P_k(\alpha_k)$ are individually Gaussian in form to derive Eqs. (14.48) and (14.50), these results evidently hold true for a considerably broader variety of field excitations.

A sketch of the Gaussian distribution function $W(\mathcal{E}, x)$ is given in Fig. 13. Since this function plays a role akin to that of a probability distribution for the complex field amplitude $\mathcal{E}$, it is evident that the absolute magnitude of the field undergoes a considerable amount of fluctuation. Thus, while the most probable value of the field amplitude is $\mathcal{E} = 0$, the amplitude will occasionally stray out

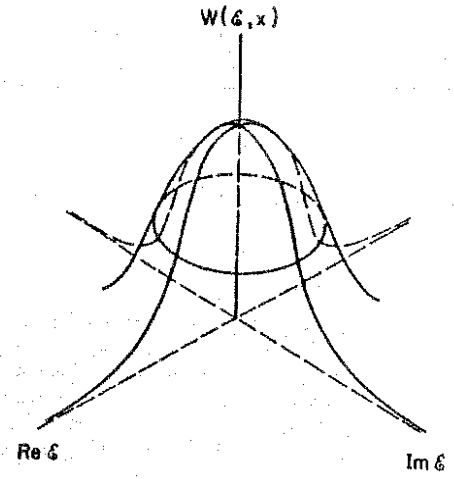

Figure 13

into the regions of the complex plane which represent the "tail" of the Gaussian and correspond to arbitrarily strong fields. The relation (14.50) between values of the correlation functions may also be stated as the relation

$$\langle|\mathcal{E}|^{2n}\rangle = n!\,\{\langle|\mathcal{E}|^2\rangle\}^n \qquad (14.51)$$

between average moments $\langle|\mathcal{E}|^l\rangle$ of the function W. The extremely rapid increase with n of the ratio $\langle|\mathcal{E}|^{2n}\rangle/\{\langle|\mathcal{E}|^2\rangle\}^n$, which the Gaussian distribution shows, is due to its "long-tailed" character.

Although the Gaussian form for the function $W(\mathcal{E}, x)$ will presumably apply almost universally to the radiation from natural or essentially chaotic sources, altogether different distributions may be required to describe the radiation from certain man-made sources. In fact the avoidance of fields which have the extremely random or noisy character of the Gaussian form of $W(\mathcal{E}, x)$ has been one of the major goals of radio-frequency technology. One of its earliest accomplishments was the development of oscillators which generate fields of extremely stable

modulus, e.g. broadcast carrier waves. These oscillators are non-linear devices and the contributions of the various mode amplitudes to the total field are not at all independently distributed as in the Gaussian case. For a stationary field generated by such an oscillator we might find the function $W(\mathcal{E}, x)$ to assume a form similar to that shown in Fig. (14); i.e., the modulus of the field, $|\mathcal{E}|$,

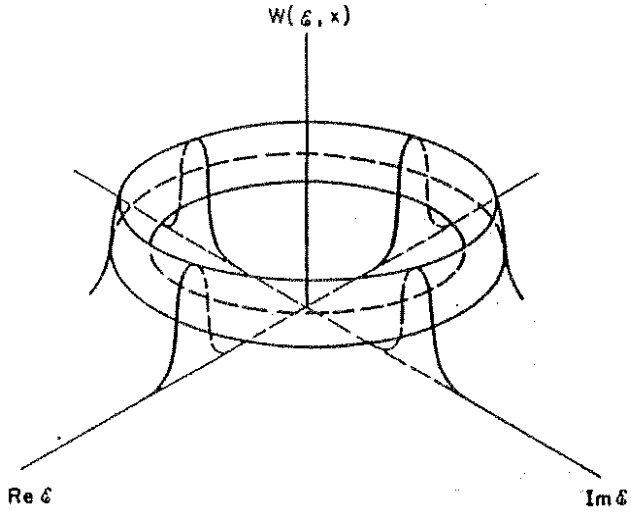

Figure 14

has only a very small probability for taking on values either appreciably smaller or larger than its root-mean-square value, $\{<|\mathcal{E}|^2>\}^{\frac{1}{2}}$.

The shape of the function $W(\mathcal{E}, x)$ furnishes an elementary insight into the origin of the photon correlation effect which was discovered by Hanbury Brown and Twiss by means of the experiment described in Lecture VIII. Let us consider the two-fold coincidence counting rate for photons when the two detectors D_1 and D_2 of Fig. 9 occupy precisely symmetrical positions relative to the half-silvered mirror m, and when the detectors are adjusted so that they register coincidences with no time delay. Since the arrangement is one in which the counters, in effect, occupy the same position and are sensitive at the same time, the coincidence rate is given by a correlation function of the form

$$G^{(2)}(x\,x,\ x\,x) = <|\mathcal{E}(x)|^4>. \tag{14.52}$$

Now, according to Eqs. (14.50) or (14.51), for all chaotic light sources we should find

$$<|\mathcal{E}(x)|^4> = 2<|\mathcal{E}(x)|^2>^2 = 2\{G^{(1)}(x,\ x)\}^2. \tag{14.53}$$

The amount by which $G^{(2)}(x\,x,\ x\,x)$ exceeds $\{G^{(1)}(x,\ x)\}^2$ is a measure of the non-random tendency of the photons to be recorded as simultaneously arriving pairs; i.e., it is a measure of the height of the "bump" on the coincidence rate curve shown in Fig. 10. Since the coincidence rate for zero time delay is twice

the background or accidental coincidence rate, the correlation effect is not a small one. (The original observations of the effect were made difficult by the relatively long response times of the counting systems compared with the time interval over which the correlation persists.)

To see the nature of the photon correlation effect for other types of distributions $W(\mathcal{E}, x)$, let us note that it is proportional to

$$G^{(2)}(x\,x,\ x\,x) - \{G^{(1)}(x,\ x)\}^2 = <|\mathcal{E}(x)|^4> - <|\mathcal{E}(x)|^2>^2 = \int W(\mathcal{E}, x)\{|\mathcal{E}|^2 - <|\mathcal{E}|^2>\}^2\, d^2\mathcal{E}. \tag{14.54}$$

One of the curious quantum mechanical properties of this expression is that, although it resembles a statistical variance for the quantity $|\mathcal{E}|^2$, it may actually take on negative as well as positive values. That is true since $W(\mathcal{E}, x)$ as we have noted, is not strictly speaking a probability distribution. It is not difficult to find states of the field for which W takes on negative values at least locally and for which the average (14.54) is consequently negative. When the field is in such states photon coincidences will be recorded with less than the random background rate by the Hanbury Brown-Twiss detection apparatus, an effect which is the reverse of the one observed for natural radiation sources.

Whenever the field is generated by an essentially classical source, i.e., one with predetermined behavior, it will be possible, as we have seen Lecture XII, to construct a P-representation for the density operator with a non-negative weight function $P(\{\alpha_k\})$. Then the function $W(\mathcal{E}, x)$ defined by Eq. (14.44) will likewise take on no negative values. We may thus state that for all classically generatable fields, the Hanbury Brown-Twiss correlation is positive,

$$G^{(2)}(x\,x,\ x\,x) - \{G^{(1)}(x,\ x)\}^2 \geq 0. \tag{14.55}$$

If the correlation effect is to vanish for fields of this type we must evidently have

$$W(\mathcal{E}, x)\{|\mathcal{E}|^2 - <|\mathcal{E}|^2>\}^2 = 0 \tag{14.56}$$

for all $\mathcal{E}$. The function $W(\mathcal{E}, x)$ can therefore only take on non-vanishing values at points lying on the circle $|\mathcal{E}|^2 = <|\mathcal{E}|^2>$. If the function $W(\mathcal{E}, x)$, in other words, is of a form which allows no amplitude modulation of the field, the correlation effect will vanish and conversely. In fact in that limit we have more generally

$$G^{(n)}(x\cdots x) = <|\mathcal{E}|^{2n}> = <|\mathcal{E}|^2>^n = \{G^{(1)}(x,\ x)\}^n \tag{14.57}$$

and all n-fold coincidence experiments show an absence of any tendency toward statistical correlations.

A number of the published discussions of the Hanbury Brown-Twiss effect explain it as being caused by the fact that photons are Bose particles and consequently have a certain tendency to cluster. That such explanations are far from complete is made evident by the fact that the quantum mechanical form of the effect may have either sign; it may constitute an anticorrelation or "replusion," rather than a positive correlation or "clumping." Furthermore the fact that classical fields have only a positive correlation effect is a clear demonstration that the average quantities one evaluates by means of the correlation functions (even where the P-representation exists) are not always equivalent in quantum theory and classical theory. The variety of fields encountered in the quantum theory is simply much larger than that allowed by classical theory.

It should be evident that the measurement of the photon correlation effect, at least at zero delay time, simply furnishes a measure of the amount of random

amplitude modulation present in fields with positive $W(\mathcal{E}, x)$. The effect should be nearly absent from the field generated by a well stabilized oscillator. In particular since a gas laser operating well above its threshold is presumably quite a stable oscillator, any Hanbury Brown-Twiss correlation found in its beam should be quite small in magnitude.

The fact that a photon correlation experiment, or its analogue in the radio-frequency region, an intensity correlation experiment, can furnish a simple way of telling whether a radiation field comes from a natrual source or a man-made one could have some interesting if rather far-fetched astronomical consequences. If intelligent beings elsewhere in the galaxy want to communicate with us, it seems reasonable to suppose that they would use amplitude-stabilized oscillators of some sort as radiators. In that case their signals, as we have seen, would have an unmistakable character even when no message was being transmitted. In fact the unmodulated signal could be easier to distinguish from background noise than the modulated one.

QUASIPROBABILITY DISTRIBUTION FOR THE FIELD AMPLITUDES AT TWO SPACE-TIME POINTS

A number of the correlation functions and other expectation values which interest us depend on the fields at two different space-time points x_1 and x_2. These averages may be expressed, when the P-representation exists, in the general form

$$\int P(\{\alpha_k\})\ \ F(\mathcal{E}(x_1\{\alpha_k\}),\ \mathcal{E}(x_2\{\alpha_k\}) \prod_k d^2\alpha_k, \tag{14.58}$$

where the function F is suitably defined for each case. Two familiar examples of such averages are the first order correlation function $G^{(1)}(x_1, x_2)$, for which we would choose

$$F = \mathcal{E}^*(x_1\{\alpha_k\})\ \mathcal{E}(x_2\{\alpha_k\}), \tag{14.59}$$

and the delayed coincidence counting rate, $G^{(2)}(x_1x_2, x_2x_1)$, for which we would choose

$$F = |\mathcal{E}(x_1\ \{\alpha_k\})|^2 \quad |\mathcal{E}(x_2\ \{\alpha_k\})|^2. \tag{14.60}$$

Now, if we define a species of distribution function $W(\mathcal{E}_1x_1, \mathcal{E}_2x_2)$, for the complex field amplitudes at the two points by means of the relation

$$W(\mathcal{E}_1x_1,\ \mathcal{E}_2x_2) = \int P(\{\alpha_k\})\,\delta^{(2)}(\mathcal{E}_1 - \mathcal{E}(x_1\{\alpha_k\}))\,\delta^{(2)}(\mathcal{E}_2 - \mathcal{E}(x_2\{\alpha_k\}))\prod_k d^2\alpha_k, \tag{14.61}$$

then an average quantity of the form (14.58) is given by the integral

$$\int W(\mathcal{E}_1x_1,\ \mathcal{E}_2x_2)\ \ F(\mathcal{E}_1, \mathcal{E}_2)\,d^2\mathcal{E}_1 d^2\mathcal{E}_2. \tag{14.62}$$

The function $W(\mathcal{E}_1x_1, \mathcal{E}_2x_2)$, more specifically, is a quasiprobability distribution which plays the same role in averaging functions of two space-time variables as the function $W(\mathcal{E}, x)$, which we discussed earlier, plays in the calculation of averages for a single space-time point. We may, in fact obtain $W(\mathcal{E}, x)$ from the two-point function by integrating over either of the field variables,

$$\begin{aligned} W(\mathcal{E}, x) &= \int W(\mathcal{E}x,\ \mathcal{E}'x')\ d^2\mathcal{E}' \\ &= \int W(\mathcal{E}'x',\ \mathcal{E}x)\ d^2\mathcal{E}'. \end{aligned} \tag{14.63}$$

When the function $P(\{\alpha_k\})$ factorizes into a product of independent weight functions, one for each mode, and when the number of excited modes is large, it is easy to show, again by techniques similar to those used in section VIII of the reprinted paper, that $W(\mathcal{E}_1x_1, \mathcal{E}_2x_2)$ assumes a Gaussian form in the two complex amplitude variables $\mathcal{E}_1$ and $\mathcal{E}_2$. To carry out the derivation we simply show that the double Fourier transform of $W(\mathcal{E}_1x_1, \mathcal{E}_2x_2)$ with respect to the amplitude variables $\mathcal{E}_1$ and $\mathcal{E}_2$ is asymptotically Gaussian in form when the number of excited modes becomes infinite. Inversion of the transform then yields a result which, for the case of stationary fields, can be written as

$$W(\mathcal{E}_1x_1, \mathcal{E}_2x_2) = \frac{1}{\pi^2 G^{(1)}(x_1x_1)G^{(1)}(x_2x_2)\{1 - |g^{(1)}(x_1x_2)|^2\}} \times$$

$$\exp\left\{-\frac{\dfrac{|\mathcal{E}_1|^2}{G^{(1)}(x_1x_1)} + \dfrac{|\mathcal{E}_2|^2}{G^{(1)}(x_2x_2)} - 2\,\mathrm{Re}\,\dfrac{\mathcal{E}_1\mathcal{E}_2^* g^{(1)}(x_1x_2)}{\{G^{(1)}(x_1x_1)G^{(1)}(x_2x_2)\}^{\frac{1}{2}}}}{1 - |g^{(1)}(x_1x_2)|^2}\right\} \tag{14.64}$$

where $g^{(1)}$ is the normalized form of the first order correlation function defined by Eq. (7.5). As a simple check of this result it is easy to verify that the average of the function (14.59) is

$$\{G^{(1)}(x_1x_1)G^{(2)}(x_2x_2)\}^{\frac{1}{2}} g^{(1)}(x_1x_2) = G^{(1)}(x_1x_2) \tag{14.65}$$

as required, and that the average of the function (14.60) is indeed

$$G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2) + |G^{(1)}(x_1,\ x_2)|^2 = G^{(2)}(x_1x_2,\ x_2x_1). \tag{14.66}$$

The function $W(\mathcal{E}_1x_1, \mathcal{E}_2x_2)$ plays a role in the theory which is analogous to that of a probability density for a compound event, i.e., finding the field $\mathcal{E}_1$ at $x_1 = (r_1, t_1)$ and $\mathcal{E}_2$ at $x_2 = (r_2, t_2)$. In probability theory it is often of interest, in dealing with such compound events, to imagine that the first part of the event has already taken place and to calculate the probability that the compound event is then completed. We may define an analogue of such a conditioned probability function by means of the relation

$$W(\mathcal{E}_1x_1|\mathcal{E}_2x_2) = \frac{W(\mathcal{E}_1x_1,\ \mathcal{E}_2x_2)}{W(\mathcal{E}_1,\ x_1)}, \tag{14.67}$$

where $W(\mathcal{E}_1,\ x_1)$ is the function defined by Eq. (14.44). The function $W(\mathcal{E}_1x_1|\mathcal{E}_2x_2)$ is analogous to a probability density for the field amplitude to have values in the neighborhood of $\mathcal{E}_2$ at $x_2 = (r_2, t_2)$, given that it had the value $\mathcal{E}_1$ at $x_1 = (r_1, t_1)$. We shall call the function the conditioned quasiprobability density; it is, strictly speaking, only measurable as a probability density in the classical or strong field limit.

When we calculate the ratio of the functions given by Eqs. (14.64) and (14.48) we find the result

$$W(\mathcal{E}_1x_1|\mathcal{E}_2x_2) = \frac{1}{\pi}\,\frac{1}{G^{(1)}(x_2x_2)\{1 - |g^{(1)}(x_1x_2)|^2\}} \times \tag{14.68}$$

$$\exp\left\{-\frac{\left|\dfrac{\mathcal{E}_2}{\{G^{(1)}(x_2x_2)\}^{\frac{1}{2}}} - \dfrac{\mathcal{E}_1}{\{G^{(1)}(x_1x_1)\}^{\frac{1}{2}}}\,g^{(1)}(x_1x_2)\right|^2}{1 - |g^{(1)}(x_1x_2)|^2}\right\}.$$